COMPLEX FUNCTIONS AND THE CAUCHY-RIEMANN EQUATIONS

Muhammadieva Maftuna
Assistant of the Mathematics Department of the
Uzbekistan-Finland Pedagogical Institute

Abstrakt. This article is about the theory of complex functions Complex functions and Cauchy-Riemann equations modern solutiondedicated to studying methods. also the article teaches students mathematical and physical thinking and Complex functions and modern methods of solving Cauchy-Riemann equations in the development of imagination studied in detail. This article is for students relatively better understanding and better results helps to achieve.

Key words: Functions, complex functions polynomial, real coefficients, complex differentiable, limit, rational function.

Enter.PQ-4708 of the President of 05.07.2020. On measures to improve the quality of education and develop scientific research in the field of mathematics. In the decision, a number of issues that have not been resolved in the field indicate the need to implement measures aimed at increasing the quality of education and the effectiveness of scientific research in the field of mathematics. Including: firstly, coherence between educational stages of mathematics teaching is not fully ensured; secondly, mathematics textbooks in general education schools contain complex issues that make it difficult to master the subject in relation to the age of the students, and are not coordinated with the topics taught in other subjects; thirdly, despite the fact that most of our talented young people who are interested in mathematics and winners of international Olympiads are from the regions, the necessary conditions for their further development in the field of higher education and science have not been created; fourthly, the relationship between scientific research in the field of mathematics and practice and production remains weak; fifthly, the relations of scientists in the field with foreign scientific and educational institutions are not enough to bring national mathematics to the world level and increase its prestige in the international community.

Of the subjectrelevance

Complex functions

In one-variable calculus, we study functions f (x) of a real variable x. Likewise, in complex analysis, we study functions f (z) of a complex variable $z \in C$ (or in some region of C). Here we expect that f (z) will in general take values in C as well. However, it will turn out that some functions are better than others. Basic examples of functions f (z) that we have already seen are: $f(z) = c$, where c is a constant (allowed to be complex),

$f(z) = z$, $f(z) = f(z) = \bar{z}$, $f(z) = Rez, f(z) = Imz$, $f(z) = |z|$, $f(z) = e^z$

The “func-tions” $f(z) = \arg \bar{z}$ $f(z) = \sqrt{z}$, and $f(z) = \log z$ are also quite interesting, but they are not well-defined (single-valued, in the terminology of complex analysis).

What is a complex valued function of a complex variable? If $z = x + iy$, then a function $f(z)$ is simply a function $F(x, y) = u(x, y) + iv(x, y)$ of the two real variables x and y. As such, it is a function (mapping) from R^2 to R^2. Here are some examples:

1. $f(z) = z$ corresponds to $F(x, y) = x + iy$ $(u = x, v = y)$;
2. $\boldsymbol{f(z) = \bar{z}}$ **,** with $F(x, y) = x - iy$ $(u = x, v = y)$;
3. $f(z) = \operatorname{Re} z$, with $F(x, y) = x$ $(u = x, v = 0$, taking values just along the real axis);

4. $f(z) = |z|$, with $F(x, y) = \sqrt{x^2 - y^2}(u = \sqrt{x^2 + y^2} v = 0$, taking values just along the real axis);

5. $f(z) = z^2$, with $F(x, y) = (x^2 - y^2) + i(2xy)$ $(u = x^2 - y^2, v = 2xy)$;
6. $f(z) = e^z$, with $F(x, y) = e^x \cos y + i(e^x \sin y)$ $(u = e^x \cos y, v = e^x \sin y)$.

If $f(z) = u + iv$, then the function $u(x, y)$ is called the real part of f and $v(x, y)$ is called the imaginary part of f. Of course, it will not in general be possible to plot the graph of f (z), which will lie in C^2 , the set of ordered pairs of complex numbers, but it is the set $\{(z, \omega \in C^2 : \omega = f(z)\}$. The graph can also be viewed as the subset of R^4 given by$\{(x, y, s, t) : s = v(x, y), t = v(x, y)\}$. In particular, it lies in a four-dimensional space.

The usual operations on complex numbers extend to complex functions: given a complex function $f(z) = u + iv$, we can define functions $\operatorname{Re} f(z) = u$,

$\operatorname{Im} f(z) = v$, $f(z) = u - iv$, $|f(z)| = u^2 + v^2$. Likewise, if $g(z)$ is another

complex function, we can define $f(z)g(z)$ and $f(z)/g(z)$ for those z for which

$g(z) \neq 0$.

Some of the most interesting examples come by using the algebraic op- erations of C. For example, a *polynomial* is an expression of the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where the a_i are complex numbers, and it defines a function in the usual way. It is easy to see that the real and imaginary parts of a polynomial $P(z)$ are polynomials in x and y. For example,

$$P(z) = (1 + i)z^2 - 3iz = (x^2 - y^2 - 2xy + 3y) + (x^2 - y^2 + 2xy - 3x)i,$$

and the real and imaginary parts of $P(z)$ are polynomials in x and y. But given two (real) polynomial functions $u(x, y)$ and $z(x, y)$, it is very rarely the case that there exists a complex polynomial $P(z)$ such that $P(z) = u + iv$. For example, it is not hard to see that x cannot be of the form $P(z)$, nor can $\bar{z}$. As we shall see later, no polynomial in x and y taking only real values for every z (i.e.v = 0) can be of the form P (z). Of course, since x = 1/2 (z + $\bar{z}$)

and y =1/ 2i (z- $\bar{z}$)every polynomial F (x, y) in x and y is also a polynomial in z and $\bar{z}$, i.e.

$$F(x, y) = Q(z, \bar{z}) = \sum_{i,j \geq 0} c_{ij} z^i z^j$$

where c_{ij} are complex coefficients.

Finally, while on the subject of polynomials, let us mention the

Fundamental Theorem of Algebra (first proved by Gauss in 1799): If $P(z)$ is a nonconstant polynomial, then $P(z)$ has a complex root. In other words, there exists a complex number c such that $P(c) = 0$. From this, it is easy to deduce the following corollaries:

1. If $P(z)$ is a polynomial of degree $n > 0$, then $P(z)$ can be factored into linear factors:

$$P(z) = a(z - c_1) \cdots (z - c_n),$$

for complex numbers a and $c_1, \ldots, c_n$.

2. Every nonconstant polynomial $p(x)$ with real coefficients can be factored into (real) polynomials of degree one or two.

Here the first statement is a consequence of the fact that c is a root of P (z) if and only if (z- c) divides P (ž), plus induction. The second statement follows from the first and the fact that, for a polynomial with **real** coefficients, complex roots occur in conjugate pairs.

One consequence of the Fundamental Theorem of Algebra is that, having enlarged the real numbers so as to have a root of the polynomial equation $x^2 + 1 = 0$, we are now miraculously able to find roots of **every** polynomial equation, including the ones where the coefficients are allowed to be complex. This suggests that it is very hard to further enlarge the complex numbers in such a way as to have any reasonable algebraic properties. Finally, we should mention that, despite its name, the Fundamental Theorem of Algebra is not really a theorem in algebra, and in fact some of the most natural proofs of this theorem are by using methods of complex function theory.

We can define a broader class of complex functions by dividing polynomi- als. By definition, a rational function R(z) is a quotient of two polynomials:

$$R(z) = P(z)/Q(z),$$

where $P(z)$ and $Q(z)$ are polynomials and $Q(z)$ is not identically zero. Using the factorization (1) above, it is not hard to see that, if $R(z)$ is not actually a polynomial, then it fails to be defined, roughly speaking, at the roots of $Q(z)$ which are not also roots of $P(z)$, and thus at finitely many points in

C. (We have to be a little careful if there are multiple roots.)

For functions of a real variable, the next class of functions we would define might be the algebraic functions, such as $\sqrt{x}$ or $\sqrt[5]{1+x^2-2\sqrt{1+x^4}}$

However, in the case of complex functions, it turns out to be fairly involved

to keep track of how to make sure these functions are well-defined (single-valued) and we shall therefore not try to discuss them here.

Finally, there are complex functions which can be defined by power series. We have already seen the most important example of such a function, $e^z = \sum_{n=0}^{\infty} z^n/n!$, which is defined for all z. Other examples are, for instance,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad |z| < 1$$

However, to make sense of such expressions, we would have to discuss con- vergence of sequences and series for complex numbers. We will not do so here, but will give a brief discussion below of limits and continuity for com- plex functions. (It turns out that, once things are set up correctly, the comparison and ratio tests work for complex power series.)

Limits and continuity

The absolute value measures the distance between two complex numbers. Thus, z_1 and z_2 are close when $|z_1 - z_2|$ is small. We can then define the *limit* of a complex function $f(z)$ as follows: we write

$$\lim_{z\to c} f(z) = L,$$

where c and L are understood to be complex numbers, if the distance from $f(z)$ to L, $|f(z) - L|$ is small whenever $|z - c|$ is small. More precisely, if we want $|f(z) - L|$ to be less than some small specified positive real number , then there should exist a positive real number δ such that, if $|z - c| < \delta$, then $|f(z) - L| < \epsilon$. Note that, as with real functions, it does not matter if $f(c) = L$ or even that $f(z)$ be defined at c. It is easy to see that, if $c = (c_1, c_2)$, $L = a + bi$ and $f(z) = u + iv$ is written as a real and an part, then $\lim_{z\to c} f(z) = L$ if and only if $\lim_{(x,y)\to(c1,c2)} u(x, y) = a$ and $\lim_{(x,y)\ (c1,c2)} v(x, y) = b$. Thus the story for limits of functions of a complex variable is the same as the story for limits of real valued functions of the variables x, y. However, a real variable x can approach a real number c only from above or below (or from the left or right, depending on your point of view), whereas there are many ways for a complex variable to approach a complex number c.

Sequences, limits of sequences, convergent series and power series can be defined similarly.

As for functions of a real variable, a function $f(z)$ is *continuous* at c if

$$\lim_{z\longrightarrow c} f(z) = f(c).$$

In other words: 1) the limit exists; 2) f (z) is defined at c; 3) its value at c is the limiting value. A function f (z) is continuous if it is continuous at all points where it is defined. It is easy to see that a function f (z) = u + iv is continuous if and only if its real and imaginary parts are continuous, and that the usual functions z, $\bar{z}$, Re z, Im z, $|z|$, e^z are continuous. (We have to be careful, though, about functions such as arg z or log z which are not well-defined.) All polynomials P (z) are continuous, as are all two-variable polynomial functions in x and y. A rational function R(z) = P (z)/Q(z) with Q(z) not identically zero is continuous where it is defined, i.e. at the finitely many points where the denominator Q(z) is not zero. More generally, if f (z) and g(z) are continuous, then so are:

1. $cf(z)$, where c is a constant;
2. $f(z) + g(z)$;
3. $f(z) \cdot g(z)$;
4. $f(z)/g(z)$, where defined (i.e. where $g(z) \neq 0$).
5. $(g \circ f)(z) = g(f(z))$, the composition of $g(z)$ and $f(z)$, where defined.

Complex derivatives

Having discussed some of the basic properties of functions, we ask now what it means for a function to have a *complex* derivative. Here we will see something quite new: this is very different from asking that its real and imaginary parts have partial derivatives with respect to x and y. We will not worry about the meaning of the derivative in terms of slope, but onlyask that the usual difference quotient exists.

Definition A function $f(z)$ is *complex differentiable* at c if

$$\lim_{z\to c}\frac{f(z)-f(c)}{z-c}$$

exists. In this case, the limit is denoted by $f'(c)$. Making the change of variable $z = c + h$, $f(z)$ is complex differentiable at c if and only if the limit

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

exists, in which case the limit is again $f''(c)$A function is complex differentiable if it is complex differentiable at every point where it is defined. For such a function f (z), the derivative defines a new function which we write as $f'(z)$ or $\frac{d}{dz}f(z)$.

For example, a constant function $f(z) = C$ is everywhere complex differ- entiable and its derivative $f'(z) = 0$. The function $f(z) = z$ is also complex differentiable, since in this case

$$\frac{f(z)-f(c)}{z-c} = \frac{z-c}{z-c} = 1.$$

Thus $(z)' = 1$. But many simple functions do not have complex derivatives. For example, consider $f(z) = \operatorname{Re} z = x$. We show that the limit

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

does not exist for any c. Let $c = a + bi$, so that $f(c) = a$. First consider $h = t$ a real number. Then $f(c + t) = a + t$ and so

$$\frac{f(c+h)-f(c)}{h} = \frac{a+t-a}{t} = 1.$$

So if the limit exists, it must be 1. On the other hand, we could use $h = it$. In this case, $f(c + it) = f(c) = a$, and

$$\frac{f(c+h)-f(c)}{h} = \frac{a-a}{it} = 0.$$

Thus approaching c along horizontal and vertical directions has given two different answers, and so the limit cannot exist. Other simple functions which can be shown not to have complex derivatives are Im z, $z\bar{z}$, and |z |.

On the bright side, the usual rules for derivatives can be checked to hold:

1. If $f(z)$ is complex differentiable, then so is $cf(z)$, where c is a constant, and $(cf(z))' = c\,f'(z)$;

2. (Sum rule) If $f(z)$ and $g(z)$ are complex differentiable, then so is $f(z)+ g(z)$, and $(f(z) + g(z))' = f'(z) + g'(z)$;

3. (Product rule) If $f(z)$ and $g(z)$ are complex differentiable, then so is $f(z)\cdot g(z)$ and $(f(z)\cdot g(z))' = f'(z)g(z) + f(z)\,g'(z)$;

4. (Quotient rule) If $f(z)$ and $g(z)$ are complex differentiable, then so is $f(z)/g(z)$, where defined (i.e. where $g(z) \neq 0$), and

$$\frac{f(z)}{g(z)} = \frac{f'(z)g(z) - f(z)\,g'(z)}{g(z)^2};$$

5. (Chain rule) If $f(z)$ and $g(z)$ are complex differentiable, then so is $f(g(z))$ where defined, and $(f(g(z))' = f'(g(z))\cdot g'(z)$.

6. (Inverse functions) If $f(z)$ is complex differentiable and one-to-one, with nonzero derivative, then the inverse function $f^{-1}(z)$ is also differentiable, and

$$(f^{-1}(z))' = 1/\,f'f^{-1}(z)).$$

Thus for example we have the power rule $(z^n)' = nz^{n-1}$, every polynomial

$P(z) = anz^n + a_{n-1}z^{n-1} + \cdots + a_0$ is complex differentiable, with

$$P'(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2}\cdots + a_1,$$

and every rational function is also complex differentiable. It follows that a function which is not complex differentiable, such as Re z or $\bar{z}$ cannot be written as a complex polynomial or rational function.

REFERENCES:

1.National personnel training program of the
Republic of Uzbekistan. www.lex.en.
2. Askarov I. B. Koziev Sh. S. Theoretical and methodological foundations of technology in creating a database of electronic educational resources .
3. Sarimsakov T.A. Theory of functions of a real variable. Tashkent: Science. 1994.
4. Sh.O. Alimov, R.R. Ashurov. Mathematical analysis. Part 1. Tashkent. Rainbow. 2012.
5. J.I. Abdullayev, R.N. Ganihojayev, M.H. Shermatov, O.I. Egamberdiyev. Functional analysis. Study guide. Tashkent-Samarkand. 2009.