TRIGONAMETRIYANING QO'SHISH FORMULALARI VA ULARNI KELTIRIB CHIQARISHNING TURLI USULLARI

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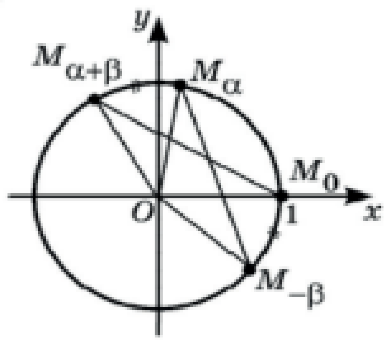
Annotatsiya: *Matematika, xususan, analiz sohasida trigonametriya muhim bo'limi juda muhim hisoblanadi. Trigonametriyada qo'shish formulalari – bu trigonametrik funksiyalarni qo'shish yoki ayirishda ishlatiladigan muhim formulalardir. Ushbu maqolada trigonametrik funksiyalarning argumentlari yig'indisi va ayirmasi uchun formulalari birlik aylanada, to'g'ri to'rtburchakda, kompleks sonlarning xossalaridan foydalanib va vektorlar orqali isbotlangan.*

Kalit so'zlar: *koinuslar yig'indisi, sinuslar yig'indisi, birlik aylana, vector, gradus, burchak.*

Trigonametriyaning asosiy formulalari sifatida yig'indining kosinusi va sinusi formulalarini keltirish mumkin, chunki ulardan qator trigonametrik formulalar kelib chiqadi. Ularni qo'shish formulalari deb ham atashadi. Bu formulalarning isboti darslikda murakkabroq usulda berilgan bo'lib, uni oson, ko'rgazmali va tushunarli qilib berish imkoniyatlari talaygina.

Quyida shu xususida va bu formulalarni matematikaning geometriya, vektorlar, koordinatalar usuli, kompleks sonlar kabi bo'limlari imkoniyatlaridan foydalanib isbotlash yo'llari haqida so'z boradi.



1-rasm

1. Keling, avval keng tarqalgan trigonametriya va koordinatalar usulidan foydalanib isbotlash yo'li haqida to'xtalaylik.

Ixtiyoriy α va β uchun

$\cos(\alpha+\beta)=cos\alpha cos\beta-sin\alpha sin\beta$ tenglik o'rinli.

Isbot: $M_0(1;0)$ nuqtani koordinata boshi atrofida $\alpha,-\beta,\alpha+\beta$, radian burchaklarga burish natijasida mos ravishda $M_\alpha, M_{-\beta}, M_{\alpha+\beta}$ nuqtalar hosil bo'ladi deylik (1-rasm).

Sinus va kosinusning ta'rifiga ko'ra, bu nuqtalar quyidagi koordinatalarga ega:

$$M_\alpha(cos\alpha; sin\alpha), M_{-\beta}(\cos(-\beta);\sin(-\beta)),$$

$$M\alpha+\beta(\cos\alpha+\beta;\sin(\alpha+\beta)).$$

$\angle M_0OM_{\alpha+\beta}=\angle M_{-\beta}OM_\alpha$ bo'lgani uchun $M_0OM_{\alpha+\beta}$ va $M_{-\beta}OM_\alpha$ teng yonli uchburchaklar teng.

Unda ularning $M_0M_{\alpha+\beta}$ va $M_{-\beta}M_\alpha$ asoslari ham teng bo'ladi:

$$(M_0M_{\alpha+\beta})^2=(M_{-\beta}M_\alpha)^2$$

Koordinatalar usuliga ko'ra ikki nuqta orasidagi masofa formulasidan foydalanib, $[1-\cos(\alpha+\beta)]^2+[sin(\alpha+\beta)]^2=[\cos(-\beta)-cos\alpha]^2+[\sin(-\beta)-sin\alpha]^2$ tenglikni hosil qilamiz.

Kvadratlarni ochib chiqib, bu tenglikni shaklini sinus funksiyaning toqligidan: $\sin(-\beta)=-\sin(\beta)$ va kosinusning juftligidan: $\cos(-\beta)=\cos(\beta)$ foydalanib almashtirib topamiz:

$$1-2\cos(\alpha+\beta)+cos^2(\alpha+\beta)+sin^2(\alpha+\beta)=$$

$$=cos^2\beta-2cos\beta cos\alpha+cos^2\alpha+sin^2\beta+2sin\beta sin\alpha+sin^2\alpha$$

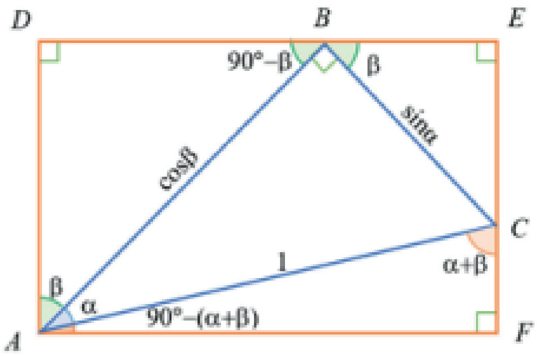
Asosiy trigonometrik ayniyatdan foydalanib,

$2 - 2\cos(\alpha+\beta) = 2 - 2cos\alpha cos\beta - 2sin\alpha sin\beta$ ni hosil qilamiz. Bundan esa quyidagi formula kelib chiqadi:

$$\cos(\alpha+\beta) = cos\alpha cos\beta - sin\alpha sin\beta \quad [1]$$

Endi bu formulani trigonometriya va geometrik yo'l bilan isbot qilaylik.

Isbot: Bitta o'tkir burchagi α ga teng, gipotenuzasi esa 1 ga teng bo'lgan to'g'ri burchakli ABC uchburchakni ADEF to'g'ri to'rtburchak ichiga rasmda ko'rsatilgandek qilib chizamiz (2-rasm)



2-rasm

$\angle BAC = \alpha$ bo'lsin. $\angle DAB = \beta$ deb belgilaymiz va ba'zi burchaklarni α va β orqali aniqlab olamiz.

ΔABDda $\angle ABD = 90^o - \beta$ ekanligini ko'rish mumkin. Unda $\angle EBC = \beta$ bo'ladi.

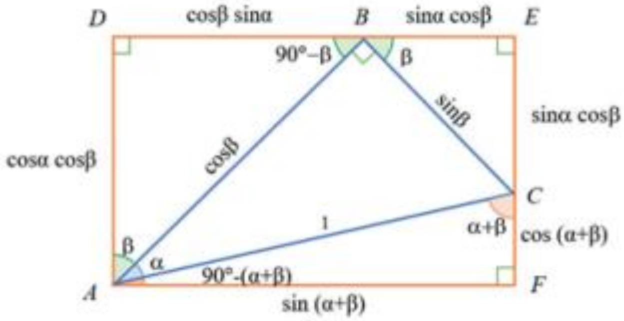
ΔCAFda $\angle CAF = 90^o - (\alpha+\beta)$ ekanligini ko'rish mumkin. Unda $\angle ACF = \alpha + \beta$ bo'ladi.

ΔABC da AC=1 bo'lgani uchun:

$$\frac{BD}{AB} = sin\alpha \text{ yoki } BC = sin\alpha,$$

$$\frac{AB}{AC} = cos\alpha \text{ yoki } AB = cos\alpha \text{ bo'ladi.}$$

Endi ADEF to'g'ri to'rtburchak tomonlari va kesmalarining uzunliklarini topamiz (3-rasm).



3-rasm

ΔABDda $AB = cos\alpha$ bo'lganligi uchun:

$$\frac{BD}{AB} = sin\beta \text{ yoki } BD = cos\alpha sin\beta, \quad (1)$$

$$\frac{AD}{AB} = cos\beta \text{ yoki } AD = cos\alpha cos\beta, \quad (2) \quad \text{bo'ladi.}$$

ΔBCE da $BC = sin\alpha$ bo'lganligi uchun:

$$\frac{CE}{BC} = sin\beta \text{ yoki } CE = sin\alpha sin\beta, \quad (3)$$

$$\frac{BE}{BC} = cos\beta \text{ yoki } BE = sin\alpha cos\beta, \quad (4) \quad \text{bo'ladi.}$$

$AC = 1$ bo'lganligi uchun:

$$\frac{CF}{AC} = \sin(\alpha+\beta) \text{ yoki } CF = \sin(\alpha+\beta), \quad (5)$$

$$\frac{AF}{AC} = \cos(\alpha+\beta) \text{ yoki } AF = \cos(\alpha+\beta), \quad (6) \quad \text{bo'ladi.}$$

ADEF to'g'ri to'rtburchakda:

a)AD=EF yoki AD=EC+CF.

Unda (2), (3) va (6) ga ko'ra

$cos\alpha cos\beta = \cos(\alpha+\beta) + sin\alpha sin\beta$ yoki $\cos(\alpha+\beta) = cos\alpha cos\beta - sin\alpha sin\beta$

b) AF=DE yoki AF=DB+BE

Unda (1), (4) va (5) ga ko'ra : $\sin(\alpha+\beta) = sin\alpha cos\beta + sin\beta cos\alpha$

Natijada yig'indining kosinusi va sinusi formulalariga ega bo'ldik. [2]

Endi qo'shish formulalarini vektorlar yordamida isbotlaymiz.

Isbot. Birlik aylanada $\vec{a} = (cos\alpha; sin\alpha), \vec{b} = (\cos(-\beta); \sin(-\beta))$

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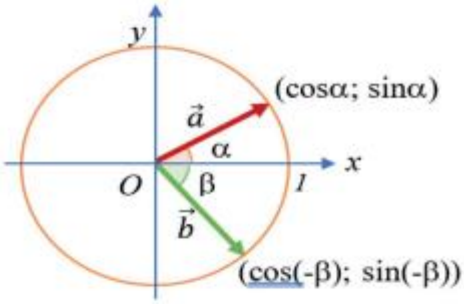
vektorlarni 4-rasmdagidek belgilab olamiz.

Bizga ma'lumki, bu vektorlarning uzunliklari: $|\vec{a}| = 1, |\vec{b}| = 1$ ular orasidagi burchak $\varphi = \alpha + \beta$ ga teng bo'ladi.



4-rasm

U holda ikki vektoring skalyar ko'paytmasi ta'rifiga ko'ra $(\vec{a}, \vec{b}) = |\vec{a}||\vec{b}|\cos(\alpha + \beta)$ ni hosil qilamiz

Ikkinchi tomondan bu ikki vektorning koordinatalaridagi skalyar ko'paytmasi

$(\vec{a}, \vec{b}) = a_1 b_1 + a_2 b_2 = = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ ga teng.

Oxirgi ikkki tenglikdan $\cos(\alpha + \beta) = cos\alpha cos\beta - sin\alpha sin\beta$ ekanligini aniqlaymiz. [3]

2. Endi esa qo'shish formulalarini kompleks sonlarning trigonometrik va darajali ko'rinishlari orasidagi quyidagi tenglikdan foydalanib isbotlab ko'ramiz:

Isbot. Qo'shish formulalarini kompleks sonlarning trigonometrik va darajali ko'rinishlar orasidagi quyidagi tenglikdan foydalanib isbotlaymiz:

$$e^{i\varphi} = cos\varphi + isin\varphi$$

Unga ko'ra

$$e^{i\alpha} = cos\alpha + isin\alpha, \quad e^{i\beta} = cos\beta + isin\beta,$$

$$e^{i(\alpha+\beta)} = cos(\alpha + \beta) + isin(\alpha + \beta) \quad (1)$$

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta} = (cos\alpha + isin\alpha)(cos\beta + isin\beta) =$$

$$= cos\alpha cos\beta + icos\alpha sin\beta + isin\alpha cos\beta - sin\alpha sin\beta =$$

$$= cos\alpha cos\beta - sin\alpha sin\beta + i(sin\alpha cos\beta + cos\alpha sin\beta) \quad (2)$$

$$cos(\alpha+\beta)+isin(\alpha+\beta)=cos\alpha cos\beta-sin\alpha sin\beta+$$
$$+i(sin\alpha cos\beta+cos\alpha sin\beta)$$

ni hosil qilamiz.

Unda bu kompleks sonlarning haqiqiy va mavhum qismlari ham teng bo'ladi:

$$\cos(\alpha+\beta)=cos\alpha cos\beta-sin\alpha sin\beta$$
$$sin(\alpha+\beta)=sin\alpha cos\beta+sin\beta cos\alpha$$

Natijada bir emas, ikkita : yig'indining kosinusi va sinusi formulalarini hosil qildik. [4]

Qo'shish formulalari yordamida trigonametriyaning qator formulalari keltirib chiqariladi.

Yig'indining kosinusi formulasidagi β o'rniga $-\beta$ ni qo'yib, kosinus funksiyaning juftligi va sinusning toqligidan foydalanib, ayirmaning kosinusi formulasi $\cos(\alpha-\beta)=cos\alpha cos\beta+sin\alpha sin\beta$ keltirib chiqariladi.

Shuningdek, bu formulalardan keltirish formulalari asosida, yig'indi va ayirmaning sinusi formulalari keltirib chiqariladi.

$$sin(\alpha+\beta)=\cos\left(\alpha+\beta-\frac{\pi}{2}\right)=\cos\alpha\cos\left(\beta-\frac{\pi}{2}\right)-sin\alpha\sin\left(\beta-\frac{\pi}{2}\right)=$$
$$=cos\alpha cos\beta+sin\alpha sin\beta$$

yoki $sin(\alpha+\beta)=sin\alpha cos\beta+sin\beta cos\alpha$

Yuqoridagi formulaga β o'rniga $-\beta$ ni qo'yib, kosinus funksiyaning juftligi va sinusning toqligidan foydalanib, ayirmaning sinusi formulasi

$$sin(\alpha-\beta)=sin\alpha cos\beta-sin\beta cos\alpha$$

Ikkilangan burchak sinusi va kosinusi formulalarini hosil qilish uchun e

sa, yig'indining sinusi va kosinusi formulalaridagi β o'rniga α ni qo'yish yetarli bo'ladi:$\cos(\alpha+\alpha)=cos\alpha cos\alpha-sin\alpha sin\alpha$ yoki $cos2\alpha=cos^2\alpha-sin^2\alpha$,

$sin(\alpha+\alpha)=sin\alpha cos\alpha+sin\alpha cos\alpha$ yoki $sin2\alpha=2sin\alpha cos\alpha$.

Shuningdek, qo'shish formulalaridan ba'zi bir bir burchak trigonometrik funksiyalarining qiymatlarini toppish uchun ham foydalansak bo'ladi.

$$cos75° = \cos(45° + 30°) = cos45°cos30° - sin45°sin30° =$$
$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4};$$
$$cos15° = \cos(45° - 30°) = cos45°cos30° + sin45°sin30° =$$
$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

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